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ON TWO STRESS CONCENTRATION PROBLEMS IN PLANE-STRESS ANISOTROPIC PLASTICITY

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Abstract—Analytical solutions are presented for the stress concentration factors at a circular hole, and at a circular inclusion, embedded in an infinite sheet which is subjected to remote uniform tension. The constitutive relations are based on a restricted version of the anisotropic flow theory proposed by Hill, along with a linear-hardening characteristic. The framework of the analysis is that of plane-stress small-strain plasticity. Some further results are derived for arbitrary hardening. It is shown that plastic anisotropy has a strong influence on the stress concentration at the hole, but is less predominant for the inclusion problem.

1. INTRODUCTION

The importance of transverse plastic orthotropy in the plane-stress analysis of metal sheets has been already recognized by Hill[1]. Further proposals of constitutive relations for the plastic behaviour of anisotropic metals have been given recently[2, 3]. An attractive suggestion by Hill[2], centres on the following definition of the effective stress σ_e:

$$2(1+R)\sigma_e^m = (1+2R)|\sigma_1 - \sigma_2|^m + |\sigma_1 + \sigma_2|^m \tag{1}$$

where (σ_1, σ_2) are the in-plane principal stresses and parameters (m, R) characterize the normal plastic anisotropy of the sheet. Equation (1) along with the associated flow rule (or deformation type theory) has been used in a number of recent studies, e.g. Refs [4–6], of plane-stress anisotropic plasticity.

In this paper we employ a restricted version of eqn (1), with $m = 1$, for the solution of two axially-symmetric stress concentration problems. Thus, in terms of the polar stress components $(\sigma_r, \sigma_\theta)$ we use the definition

$$2(1+R)\sigma_e = (1+2R)|\sigma_r - \sigma_\theta| + |\sigma_r + \sigma_\theta|. \tag{2}$$

The variation of that yield locus with parameter R is shown in Fig. 1. Note that the isotropic Tresca locus is formed by the lines of $R = \infty$ in the quadrants where $(\sigma_r, \sigma_\theta)$ have opposite signs, and by the lines $R = 0$ in the quadrants where $(\sigma_r, \sigma_\theta)$ have identical signs.

Within the usual assumptions of the elasto-plastic flow theory we obtain from eqn (2) the three constitutive relations, for the total strain rates, namely

$$\dot{\varepsilon}_r = \frac{1}{E}(\dot{\sigma}_r - \nu\dot{\sigma}_\theta) + [(1+2R)\operatorname{sgn}(\sigma_r - \sigma_\theta) + \operatorname{sgn}(\sigma_r + \sigma_\theta)]\frac{\dot{\varepsilon}_P}{2(1+R)} \tag{3}$$

$$\dot{\varepsilon}_\theta = \frac{1}{E}(\dot{\sigma}_\theta - \nu\dot{\sigma}_r) + [-(1+2R)\operatorname{sgn}(\sigma_r - \sigma_\theta) + \operatorname{sgn}(\sigma_r + \sigma_\theta)]\frac{\dot{\varepsilon}_P}{2(1+R)} \tag{4}$$

$$\dot{\varepsilon}_z = -\frac{\nu}{E}(\dot{\sigma}_r + \dot{\sigma}_\theta) - [\operatorname{sgn}(\sigma_r + \sigma_\theta)]\frac{\dot{\varepsilon}_P}{1+R} \tag{5}$$

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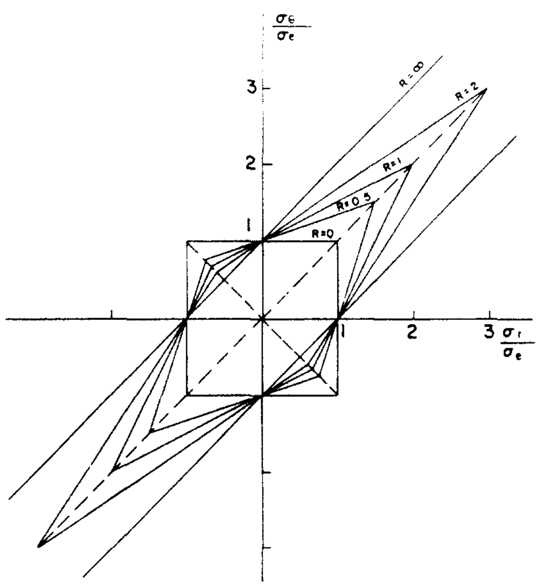


Fig. 1. Variation of the yield locus with parameter R.

where (ν, E) are the elastic constants, and ε_P is the effective plastic strain and a known function of σ_e.

When the respective signs of $(\sigma_r - \sigma_\theta)$ and $(\sigma_r + \sigma_\theta)$ remain constant along the stress path, as happens in the problems investigated here, it becomes possible to integrate eqns (3)–(5) over the loading history. The flow theory thus coincides with the deformation theory regardless of any particular hardening characteristic.

The problems considered are those of an infinite sheet, subjected to remote uniform tension, which contains a hole or a rigid inclusion at the centre. The analysis is within the framework of plane-stress small-strain plasticity. A further simplification is gained by employing a linear-hardening response function which, in turn, leads to linear stress–strain relations in the plastic range. We derive, however, the equations that govern the solutions, of the two problems, for arbitrary hardening.

Analytical treatments of the small-strain hole problem have been given by Budiansky[3, 7], while numerical investigations are presented in Refs [4, 5, 8–10]. Much less, however, has apparently been done on the inclusion problem[10, 11].

The present analysis, with the linear-hardening model, results in transcendental equations for the stress concentration factors. The latter is thus easily determined as a function of applied load, hardening parameter and degree of anisotropy. Solutions for the Tresca material are obtained as special cases. The results indicate a strong influence of the coupling between the load and material anisotropy on the stress concentration factor. The hole problem, however, appears to be more sensitive, to these parameters, in comparison with the rigid inclusion problem.

2. STRESS CONCENTRATION AT A CIRCULAR HOLE

Consider a circular hole, of radius a, embedded in an infinite sheet subjected to uniform tension σ_∞ at infinity. Let r denote the radial coordinate and let $\rho = r/a$ denote the corresponding non-dimensional coordinate. The boundary conditions are therefore

$$\sigma_r(\rho = 1) = 0, \qquad \sigma_r(\rho = \infty) = \sigma_\infty. \tag{6}$$

For sufficiently small values of σ_∞ the sheet will deform elastically with a stress concentration factor of

$$k = \frac{\sigma_\theta(\rho = 1)}{\sigma_\infty} = 2. \tag{7}$$

Yielding will begin at the hole when the applied load reaches the value

$$S = \frac{1}{2} \quad \text{with} \quad S = \frac{\sigma_\infty}{Y} \tag{8}$$

where Y is the uniaxial yield stress. Relation (8) follows from eqn (2) with the stresses expressed by the known elastic solution.

With a further increase of σ_∞, beyond eqn (8), a plastic zone will develop from the boundary $\rho = 1$ up to the elasto-plastic interface at $\rho = \rho_i$. Within that zone we have $\sigma_\theta > \sigma_r > 0$ so that the constitutive equations, eqns (3)–(5), can be integrated along the loading path, namely

$$\varepsilon_r = \Sigma_r - \nu\Sigma_\theta - s\varepsilon_P \tag{9}$$

$$\varepsilon_\theta = \Sigma_\theta - \nu\Sigma_r + \varepsilon_P \tag{10}$$

$$\varepsilon_z = -\nu(\Sigma_r + \Sigma_\theta) - (1 - s)\varepsilon_P \tag{11}$$

where $(\Sigma_r, \Sigma_\theta)$ are the non-dimensionalized (with respect to the elastic modulus E) stress components and

$$s = \frac{R}{1 + R}. \tag{12}$$

The effective stress, eqn (2), can now be written in the dimensionless form

$$\Sigma = \Sigma_\theta - s\Sigma_r \tag{13}$$

with the isotropic Tresca definition when $s = 0$.

For an elastic/linear-hardening solid we have the stress–strain characteristic

$$\varepsilon_P = N(\Sigma - \Sigma_Y) \tag{14}$$

where

$$N = \frac{1 - \eta}{\eta}, \qquad \eta = \frac{E_T}{E}, \qquad \Sigma_Y = \frac{Y}{E} \tag{15}$$

and E_T is the tangent modulus.

Combining eqn (13) with eqn (14) we find that eqns (9) and (10) become

$$\varepsilon_r = (1 + Ns^2)\Sigma_r - (\nu + Ns)\Sigma_\theta + Ns\Sigma_Y \tag{16}$$

$$\varepsilon_\theta = \frac{1}{\eta}\Sigma_\theta - (\nu + Ns)\Sigma_r - N\Sigma_Y. \tag{17}$$

Now, from the geometrical relations $\varepsilon_r = \mathrm{d}u/\mathrm{d}r$ and $\varepsilon_\theta = u/r$, where u is the radial displacement, we get the compatibility equation

$$\rho\frac{\mathrm{d}\varepsilon_\theta}{\mathrm{d}\rho} + \varepsilon_\theta - \varepsilon_r = 0. \tag{18}$$

Inserting the strains (16) and (17) in eqn (18), and using the equilibrium equation

$$\rho \frac{d\Sigma_r}{d\rho} + \Sigma_r - \Sigma_\theta = 0 \tag{19}$$

gives

$$\rho \frac{d\Sigma_\theta}{d\rho} + \Sigma_\theta - [\eta + (1-\eta)s^2]\Sigma_r = (1-\eta)(1+s)\Sigma_Y. \tag{20}$$

The solution of the two linear differential equations, eqns (19) and (20), with $(\Sigma_r, \Sigma_\theta)$ as unknowns, is readily found as

$$\Sigma_r = C_1\rho^{\lambda_1} + C_2\rho^{\lambda_2} + \frac{\Sigma_Y}{1-s} \tag{21}$$

$$\Sigma_\theta = \phi C_1\rho^{\lambda_1} - \phi C_2\rho^{\lambda_2} + \frac{\Sigma_Y}{1-s} \tag{22}$$

where

$$\phi = \sqrt{(\eta + (1-\eta)s^2)} \tag{23}$$

and

$$\lambda_1 = -1 + \phi, \qquad \lambda_2 = -1 - \phi. \tag{24}$$

Note that both λ_1 and λ_2 are real and negative.

This solution holds for $1 \leqslant \rho \leqslant \rho_i$. For $\rho \geqslant \rho_i$ we have the elastic solution, which satisfies the stress condition at infinity

$$\Sigma_r = T - \frac{A}{\rho^2}, \qquad \Sigma_\theta = T + \frac{A}{\rho^2}, \quad \text{with } T = \frac{\sigma_\infty}{E}. \tag{25}$$

The complete solution for the stress field contains four constants (C_1, C_2, ρ_i, A) which should be determined. At the interface we require continuity of the normal stress Σ_r or, from eqn (21) and the first of eqns (25)

$$C_1\rho_i^{\lambda_1} + C_2\rho_i^{\lambda_2} + \frac{\Sigma_Y}{1-s} = T - A\rho_i^{-2}. \tag{26}$$

Also, at the interface, the effective stress Σ should be equal to the yield stress Σ_Y. Thus, combining eqns (21) and (22) with eqn (13) gives

$$(\phi - s)C_1\rho_i^{\lambda_1} - (\phi + s)C_2\rho_i^{\lambda_2} = 0 \tag{27}$$

while from eqns (25) and (13) we get

$$(1-s)T + (1+s)A\rho_i^{-2} = \Sigma_Y. \tag{28}$$

It is worth mentioning that eqns (26)–(28) assure the continuity of the radial displacement across the elasto-plastic interface.

Finally, we have the free boundary condition at $\rho = 1$ which can be written, with the aid of eqn (21), as

$$C_1 + C_2 + \frac{\Sigma_Y}{1 - s} = 0. \tag{29}$$

The solution of eqns (26)–(29) determines the stress field. To this end we eliminate $A\rho_i^{-2}$ from eqns (26) and (28) and combine the resulting equation with eqn (27), thus obtaining

$$C_1 = \frac{\phi + s}{(1 + s)\phi}\left(T - \frac{\Sigma_Y}{1 - s}\right)\rho_i^{-\lambda_1} \tag{30}$$

$$C_2 = \frac{\phi - s}{(1 + s)\phi}\left(T - \frac{\Sigma_Y}{1 - s}\right)\rho_i^{-\lambda_2}. \tag{31}$$

A further substitution of these relations in eqn (29) gives

$$(\phi + s)\rho_i^{-\lambda_1} + (\phi - s)\rho_i^{-\lambda_2} = \frac{(1 + s)\phi}{1 - (1 - s)S}. \tag{32}$$

Equation (32) determines the location of the elasto-plastic interface as a function of the applied load. Note that at the onset of yielding $S = 1/2$ and $\rho_i = 1$ as expected. Also, when $(1 - s)S \to 1$ the whole sheet becomes plastic with $\rho_i \to \infty$. The solution is therefore valid only until the complete yielding load given, with the aid of eqn (12), by

$$(1 - s)S = 1 \quad \text{or} \quad \sigma_\infty = (1 + R)Y. \tag{33}$$

Once the dependence of ρ_i on σ_∞ has been found, it is possible to calculate constants C_1, C_2 from eqns (30) and (31), and constant A from eqn (28). The stress concentration factor can then be found from the relation $\Sigma_\theta(\rho = 1) = kT$ or, from eqn (22)

$$\phi(C_1 - C_2) = kT - \frac{\Sigma_Y}{1 - s}. \tag{34}$$

Alternatively, one can arrive at a single equation for k; inserting eqns (30) and (31) in eqn (34) gives

$$(\phi + s)\rho_i^{-\lambda_1} - (\phi - s)\rho_i^{-\lambda_2} = (1 + s)\frac{1 - (1 - s)kS}{1 - (1 - s)S}. \tag{35}$$

Eliminating ρ_i from eqns (32) and (35) leads to the desired equation for k

$$[1 - (1 - s)S]\left[1 + \left(\frac{1 - s}{\phi + s}\right)(1 - kS)\right]^{\lambda_2/2\phi} - \left(\frac{1 + s}{2}\right)\left[1 - \left(\frac{1 - s}{\phi - s}\right)(1 - kS)\right]^{\lambda_1/2\phi} = 0 \tag{36}$$

with the classical solution of $k = 2$ at $S = 1/2$.

It is instructive to investigate the asymptotic behaviour of the solution as the load at infinity approaches eqn (33). Since both λ_1, λ_2 are negative, and $\lambda_2 < \lambda_1$, we find from eqn (32) that near the complete yielding load

$$[1 - (1 - s)S]\rho_i^{-\lambda_2} \sim \frac{(1 + s)\phi}{\phi - s}. \tag{37}$$

Combining this expression with eqn (35) shows that asymptotically the stress concentration factor increases with increasing load

$$k \sim 1 + \phi - \frac{\phi(\phi + s)}{\phi - s}\rho_i^{-2\phi} \tag{38}$$

and

$$k = 1 + \phi = 1 + \sqrt{(\eta + (1 - \eta)s^2)} \tag{39}$$

when the whole sheet becomes plastic. Relation (39) follows also directly from eqn (36). For the Tresca material it reads $k = 1 + \sqrt{\eta}$.

For load levels above eqn (33) a new plastic zone will develop, starting at infinity and spreading inwards as the load increases. That zone is characterized by an equibiaxial state of stress where $\sigma_r = \sigma_\theta = \sigma_\infty$ or, in non-dimensional variables

$$\Sigma_r = \Sigma_\theta = T \quad \text{for} \quad \rho_b \leqslant \rho \leqslant \infty \tag{40}$$

where ρ_b stands for the interface between the plastic zone and the equibiaxial stress zone. Solution (21) and (22) is still valid within the plastic zone $1 \leqslant \rho \leqslant \rho_b$, and so we can write the two interface conditions

$$C_1\rho_b^{\lambda_1} + C_2\rho_b^{\lambda_2} = T - \frac{\Sigma_Y}{1 - s} \tag{41}$$

$$(1 - \phi)C_1\rho_b^{\lambda_1} + (1 + \phi)C_2\rho_b^{\lambda_2} = 0 \tag{42}$$

for radial stress continuity and equality of the two stress components, respectively. Eliminating C_1 and C_2 from eqns (41) and (42) and substituting the result in the free boundary condition (29) gives

$$(1 + \phi)\rho_b^{-\lambda_1} - (1 - \phi)\rho_b^{-\lambda_2} = -\frac{2\phi}{(1 - s)S - 1} \tag{43}$$

which, like eqn (32), determines the location of the interface ρ_b. A further substitution of constants C_1, C_2 from eqns (41) and (42) in eqn (34) results in

$$(1 + \phi)\rho_b^{-\lambda_1} + (1 - \phi)\rho_b^{-\lambda_2} = 2\frac{(1 - s)kS - 1}{(1 - s)S - 1}. \tag{44}$$

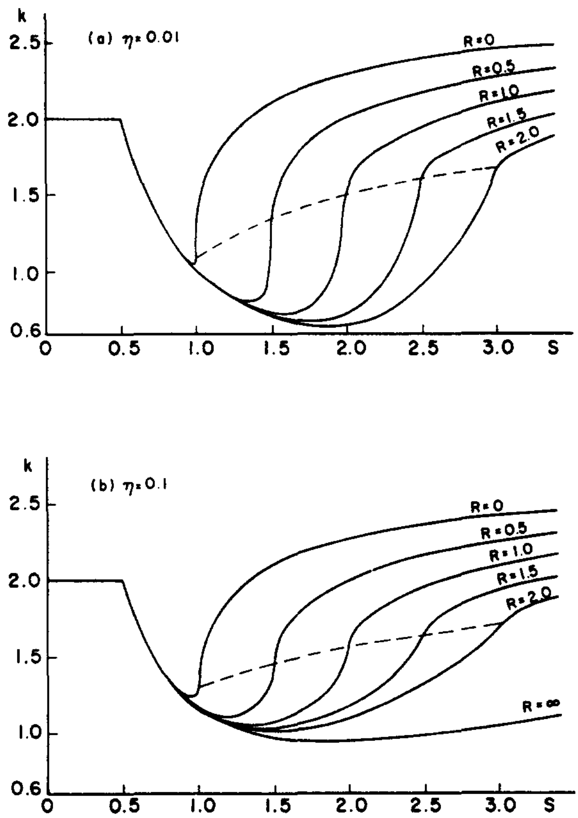


Fig. 2. Stress concentration factors at a circular hole. The broken line indicates complete yielding of the sheet. The Tresca solid is described by $R = 0$.

Eliminating ρ_b from eqns (43) and (44) we get a single equation for the stress concentration factor, namely

$$[(1-s)S-1]\left[\left(\frac{1-s}{1+\phi}\right)kS-1\right]^{\lambda_2/2\phi}-\left[\left(\frac{1-s}{1-\phi}\right)kS-1\right]^{\lambda_1/2\phi}=0 \tag{45}$$

in agreement with eqn (39) at $(1-s)S = 1$.

There is little practical importance of the present solution at very high loads since finite deformations should then be accounted for. It is interesting, however, to note that as σ_∞ becomes very large the stress concentration factor k as well as the interface ρ_b approach the finite values

$$k=\sqrt{(1-\phi^2)}\left(\frac{1+\phi}{1-\phi}\right)^{1/2\phi} \tag{46}$$

$$\rho_b=\left(\frac{1+\phi}{1-\phi}\right)^{1/2\phi}. \tag{47}$$

These relations follow from a simple asymptotic analysis of eqns (43)–(45).

Figure 2 displays the dependence of the stress concentration factor on the load at infinity, for several values of parameter R and for two representative hardening parameters, as obtained from eqns (36) and (45). Also shown is the complete yielding locus which follows from eqn (39), with $(1-s)S = 1$, as

$$k=1+\sqrt{(\eta+(1-\eta)(1-1/S)^2)}. \tag{48}$$

The initial behaviour of the curves in Fig. 2 is close to the elastic/perfectly-plastic solution $kS = 1$, but with increasing load-hardening and anisotropy effects become predominant. Note in particular the sensitivity of the stress concentration factor to parameter R.

Since the validity of our solution is restricted to small strains, it is worthwhile to consider the amount of deformation at the hole. The effective stress at the boundary is $k\sigma_\infty$ so that the effective strain ε_a, which is equal to $\varepsilon_\theta(\rho = 1)$, is, from eqn (17)

$$\varepsilon_a = \frac{k}{\eta}T - N\Sigma_Y. \tag{49}$$

Recalling that the yield strain ε_Y is equal to Σ_Y we get a suitable measure of the plastification at the hole by

$$\frac{\varepsilon_a}{\varepsilon_Y} = \frac{k}{\eta}S - N. \tag{50}$$

Thus, for example, at complete yielding this relation gives, with the aid of eqn (39)

$$\frac{\varepsilon_a}{\varepsilon_Y} = 1 + \frac{\phi + s}{\eta(1 - s)}. \tag{51}$$

For the curves in Fig. 2(a), with $\eta = 0.01$, that ratio is in the range of 10–400 for R ranging between 0 and 2. The corresponding range for Fig. 2(b), with a larger amount of hardening $\eta = 0.1$, is 4–40. Normally ε_Y is about 2×10^{-3} so that one can expect the small strain limit of validity to hold up to values of $\varepsilon_a/\varepsilon_Y \approx 75$.

While the analysis presented so far is based on the elastic/linear-hardening characteristic, it is worth mentioning that it is possible to derive a simple differential equation for arbitrary hardening functions. To this end we combine eqns (9) and (10) with eqn (13) and eliminate Σ_θ from the stress–strain relations, i.e.

$$\varepsilon_r = (1 - \nu s)\Sigma_r - \nu\Sigma - s\varepsilon_P \tag{52}$$

$$\varepsilon_\theta = (s - \nu)\Sigma_r + \varepsilon \tag{53}$$

where $\varepsilon = \Sigma + \varepsilon_P$ is the total strain and a known function of Σ. Next we eliminate $d\rho/\rho$ from eqns (18) and (19) and substitute eqns (52) and (53) in the resulting relation. This leads directly to the equation

$$(1 + s)\frac{d\Sigma_r}{d\varepsilon} = -\frac{\Sigma - (1 - s)\Sigma_r}{\varepsilon - (1 - s)\Sigma_r} \tag{54}$$

along with the boundary conditions $\Sigma_r = 0$ when $\Sigma = kT$ and $\Sigma_r = T$ when $\Sigma = (1 - s)T$. A straightforward numerical solution determines the value of k (as a function of σ_∞). For a continuous ε–Σ curve, with no definite yield point, there is of course no need to consider an elasto-plastic interface, but it might be necessary to shift to Σ as the independent variable (say with the Ramberg–Osgood characteristic).

When $R \to \infty$ and $s \to 1$, we obtain from eqn (54) the simple relation $2d\Sigma_r/d\varepsilon = -\Sigma/\varepsilon$ or, after integrating

$$\int_0^{kT} \frac{E_s}{E_T} d\Sigma = 2T \tag{55}$$

where E_s and E_T are the secant and tangent moduli, respectively.

As a simple application of eqn (55) consider the modification of the Ramberg–Osgood relation, proposed in Ref. [7], where

$$\varepsilon = \Sigma \quad \text{for}\, \Sigma \leqslant \Sigma_Y, \qquad \varepsilon = \Sigma_Y \left(\frac{\Sigma}{\Sigma_Y}\right)^n \quad \text{for}\, \Sigma \geqslant \Sigma_Y. \tag{56}$$

The stress concentration factor is readily found as

$$k = \frac{2}{n} + \left(1 - \frac{1}{n}\right)\frac{1}{S} \quad \text{for}\, S \geqslant \frac{1}{2} \tag{57}$$

with the asymptotic value $2/n$ derived earlier in Ref. [3].

For an elastic/linear-hardening material eqn (54) reduces to a homogeneous equation and can be integrated to yield relation (36), but the subsequent analysis is more complicated than the one presented here. On the other hand, for $R \to \infty$ it is straightforward to use eqn (55) with the resulting equation

$$\ln\left(\frac{k}{\eta}S - N\right) - \left(\frac{2-k}{1-\eta}\right)S = 0 \quad \text{for}\, S \geqslant \frac{1}{2}. \tag{58}$$

Values of the stress concentration factor evaluated from this equation are shown in Fig. 2(b) for $\eta = 0.1$.

A further simplification of eqn (54) is possible for rigid/plastic materials, where $E \to \infty$, with arbitrary hardening. The governing equation can be written as

$$(1+s)\varepsilon\frac{\mathrm{d}\sigma_r}{\mathrm{d}\varepsilon} - (1-s)\sigma_r = -\sigma_e \tag{59}$$

where ε is identified here with the total effective plastic strain ε_P. The solution of eqn (59) is

$$\sigma_r = \frac{\varepsilon^{(1-s)/(1+s)}}{1+s}\int_{\varepsilon}^{\varepsilon_a} \sigma_e \varepsilon^{-2/(1+s)}\,\mathrm{d}\varepsilon \tag{60}$$

and the stress concentration factor is obtained from the condition, at infinity

$$\sigma_\infty = \frac{\varepsilon_\infty^{(1-s)/(1+s)}}{1+s}\int_{\varepsilon_\infty}^{\varepsilon_a} \sigma_e \varepsilon^{-2/(1+s)}\,\mathrm{d}\varepsilon. \tag{61}$$

The value of this restricted analysis lies in its asymptotic nature for the deep plastic range. Thus, for example, with the pure power characteristic $\sigma_e = \sigma_0 \varepsilon^{1/n}$ we recover from eqn (60) the expression

$$k = \left(\frac{1}{1+R}\right)\left(\frac{n}{1+2R}\right)^{(1+2R)/(n-1-2R)} \tag{62}$$

derived by Budiansky[3]. Note that for this particular model the boundary values associated with eqn (61) are

$$\varepsilon_\infty = \frac{(1-s)^n \sigma_\infty^n}{\sigma_0^n}, \qquad \varepsilon_a = \frac{k^n \sigma_\infty^n}{\sigma_0^n}.$$

3. STRESS CONCENTRATION AT A RIGID CIRCULAR INCLUSION

The boundary conditions for this problem are

$$u(\rho = 1) = 0, \qquad \sigma_r(\rho = \infty) = \sigma_\infty. \tag{63}$$

Before yielding the stress field is given by eqn (25) with

$$A = -\left(\frac{1-\nu}{1+\nu}\right)T. \tag{64}$$

Thus $\Sigma_r > \Sigma_\theta > 0$ and the effective stress, eqn (2), becomes

$$\Sigma = \Sigma_r - s\Sigma_\theta \tag{65}$$

which agrees with the Tresca definition with $s = 0$.

Yielding will begin at the inclusion when

$$\sigma_\infty = \frac{1+\nu}{2(1-\nu s)}Y. \tag{66}$$

The stress concentration factor is defined here as

$$k = \frac{\sigma_e(\rho = 1)}{\sigma_\infty} \tag{67}$$

so that during the complete elastic deformation

$$k = \frac{2(1-\nu s)}{1+\nu}. \tag{68}$$

Note that definition (67) agrees with eqn (7) since in the hole problem σ_θ is identical with the effective stress at the hole.

After the onset of yielding a plastic zone will spread from the inclusion and increase as the load is raised. Let ρ_i denote again the elasto-plastic interface. Expecting that $\Sigma_r > \Sigma_\theta > 0$ within the plastic zone $1 \leqslant \rho \leqslant \rho_i$, we can integrate the constitutive equations, eqns (3)–(5), and put them in the form

$$\varepsilon_r = \Sigma_r - \nu\Sigma_\theta + \varepsilon_P \tag{69}$$

$$\varepsilon_\theta = \Sigma_\theta - \nu\Sigma_r - s\varepsilon_P \tag{70}$$

$$\varepsilon_z = -\nu(\Sigma_r + \Sigma_\theta) - (1-s)\varepsilon_P. \tag{71}$$

Combining expression (14) for the linear-hardening model with eqn (65) we find that eqns (69) and (70) can be rewritten as

$$\varepsilon_r = \frac{1}{\eta}\Sigma_r - (\nu + Ns)\Sigma_\theta - N\Sigma_Y \tag{72}$$

$$\varepsilon_\theta = -(\nu + Ns)\Sigma_r + (1 + Ns^2)\Sigma_\theta + Ns\Sigma_Y. \tag{73}$$

Inserting eqns (72) and (73) in the compatibility eqn (18) gives, with the aid of the equilibrium equation

$$\phi^2\left(\rho\frac{d\Sigma_\theta}{d\rho} + \Sigma_\theta\right) - \Sigma_r = -(1-\eta)(1+s)\Sigma_Y. \tag{74}$$

The solution of the pair (74) and (19) is

$$\Sigma_r = C_1\rho^{-\lambda_1/\phi} + C_2\rho^{\lambda_2/\phi} + \frac{\Sigma_Y}{1-s} \tag{75}$$

$$\Sigma_\theta = \frac{1}{\phi}(C_1\rho^{-\lambda_1/\phi} - C_2\rho^{\lambda_2/\phi}) + \frac{\Sigma_Y}{1-s} \tag{76}$$

where ϕ and λ_1, λ_2 are given by eqns (23) and (24), respectively.

The stress field within the elastic zone $\rho_i \leqslant \rho \leqslant \infty$ is still given by eqn (25). The interface conditions are therefore

$$C_1\rho_i^{-\lambda_1/\phi} + C_2\rho_i^{\lambda_2/\phi} + \frac{\Sigma_Y}{1-s} = T - A\rho_i^{-2} \tag{77}$$

$$(\phi - s)C_1\rho_i^{-\lambda_1/\phi} + (\phi + s)C_2\rho_i^{\lambda_2/\phi} = 0 \tag{78}$$

$$(1-s)T - (1+s)A\rho_i^{-2} = \Sigma_Y \tag{79}$$

expressing radial stress continuity and yielding at $\rho = \rho_i$. A fourth condition is obtained from the requirement that the radial displacement should vanish at $\rho = 1$ or, from eqns (73), (75) and (76)

$$\Gamma C_1 - \Delta C_2 = -\left(\frac{1-\nu}{1-s}\right)\eta\,\Sigma_Y \tag{80a}$$

where

$$\Gamma = \phi - \nu\eta - (1-\eta)s \qquad \Delta = \phi + \nu\eta + (1-\eta)s. \tag{80b}$$

The solution of eqns (77)–(80a) can be cast into the form

$$C_1 = \frac{\phi + s}{1+s}\left(T - \frac{\Sigma_Y}{1-s}\right)\rho_i^{\lambda_1/\phi} \tag{81}$$

$$C_2 = -\frac{\phi - s}{1+s}\left(T - \frac{\Sigma_Y}{1-s}\right)\rho_i^{-\lambda_2/\phi} \tag{82}$$

$$(\phi + s)\Gamma\rho_i^{\lambda_1/\phi} + (\phi - s)\Delta\rho_i^{-\lambda_2/\phi} = \frac{\eta(1-\nu)(1+s)}{1-(1-s)S} \tag{83}$$

where the last equation gives the interface location as a function of the load. Constants C_1, C_2 are then determined, while constant A can be found from eqn (79). This solution holds until complete yielding of the sheet (when $\rho_i \to \infty$), as the load approaches the value (33).

From definition (67), with the aid of eqns (65), (75) and (76), we get the additional relation

$$(\phi - s)C_1 + (\phi + s)C_2 = \phi(kT - \Sigma_Y) \tag{84}$$

or, after substituting eqns (81) and (82)

$$\rho_i^{\lambda_1/\phi} - \rho_i^{-\lambda_2/\phi} = \left(\frac{\phi}{\eta}\right)\frac{1 - kS}{1 - (1 - s)S}. \tag{85}$$

Once the dependence of ρ_i on S has been established, by eqn (83), it is possible to find the stress concentration factor from eqn (85). Alternatively, we can eliminate ρ_i from eqns (83) and (85) and write a single equation relating k to the load at infinity, namely

$$[1 - (1 - s)S]\left[1 + \frac{\phi(\phi - s)\Delta}{\eta^2(1 - \nu)(1 + s)}(1 - kS)\right]^{\lambda_2/2} - \frac{(1 - \nu)(1 + s)}{2(1 - \nu s)}\left[1 - \frac{\phi(\phi + s)\Gamma}{\eta^2(1 - \nu)(1 + s)}(1 - kS)\right]^{-\lambda_1/2} = 0. \tag{86}$$

At the onset of yielding, eqn (66), we recover from eqn (86) the stress concentration given by eqn (68).

The complete yielding load is again given by $(1 - s)S = 1$ and a simple asymptotic analysis shows that near this load

$$[1 - (1 - s)S]\,\rho_i^{-\lambda_2/\phi} \sim \frac{\eta(1 - \nu)(1 + s)}{(\phi - s)\Delta} \tag{87}$$

and at full yielding

$$k = 1 - s + \frac{\eta^2(1 - \nu)(1 - s^2)}{\phi(\phi - s)\Delta}. \tag{88}$$

For the Tresca material the value of k reduces to

$$k = \frac{1 + \sqrt{\eta}}{1 + \nu\sqrt{\eta}}. \tag{89}$$

When σ_∞ increases beyond $(1 + R)Y$ a second plastic zone will develop, starting at infinity and spreading inwards. The state of stress within that zone is again equibiaxial, eqn (40), and the interface conditions read

$$C_1\rho_b^{-\lambda_1/\phi} + C_2\rho_b^{\lambda_2/\phi} = T - \frac{\Sigma_Y}{1 - s} \tag{90}$$

$$(1 - \phi)C_1\rho_b^{-\lambda_1/\phi} - (1 + \phi)C_2\rho_b^{\lambda_2/\phi} = 0. \tag{91}$$

Combining these two relations with the rigid boundary condition (80a) gives

$$(1 + \phi)\Gamma\rho_b^{\lambda_1/\phi} - (1 - \phi)\Delta\rho_b^{-\lambda_2/\phi} = -\frac{2\eta(1 - \nu)}{(1 - s)S - 1}. \tag{92}$$

Similarly, from eqns (84), (90) and (91) we find that

$$(1 + \phi)(\phi - s)\rho_b^{\lambda_1/\phi} + (1 - \phi)(\phi + s)\rho_b^{-\lambda_2/\phi} = 2\phi(1 - s)\frac{kS - 1}{(1 - s)S - 1}. \tag{93}$$

Equation (92) determines the relation between the load and the interface location. The value of the corresponding stress concentration factor can then be obtained from eqn (93). Alternatively, we can eliminate ρ_b from eqns (92) and (93) and write a single equation relating k to the applied load, namely

$$[(1 - s)S - 1]\left[\Delta(kS - 1) - \frac{\eta(1 - \nu)(\phi + s)}{\phi(1 - s)}\right]^{\lambda_2/2} - \frac{1}{\sqrt{(1 - \phi^2)}}\left(\frac{1 - \phi}{1 + \phi}\right)^{\phi/2}\left[\frac{\phi(1 - s)}{\eta(1 - \nu s)}\right]\left[\Gamma(kS - 1) + \frac{\eta(1 - \nu)(\phi - s)}{\phi(1 - s)}\right]^{-\lambda_1/2} = 0. \tag{94}$$

The solution of this equation at the complete yielding load coincides with eqn (88). As S becomes very large we find the asymptotic values

$$k = \frac{\eta}{\phi}(1 - \nu s)\sqrt{(1 - \phi^2)}\left(\frac{1 + \phi}{1 - \phi}\right)^{\phi/2}\sqrt{(\Gamma^{\lambda_1}\Delta^{\lambda_2})} \tag{95}$$

$$\rho_b = \left[\frac{(1 + \phi)\Gamma}{(1 - \phi)\Delta}\right]^{\phi/2}. \tag{96}$$

Relation (96), however, is applicable only as long as $\rho_b > 1$ or, with the aid of eqns (80b), in the range of

$$s(1 - s) < \frac{1 - \nu}{N}. \tag{97}$$

If the material parameters do not satisfy this inequality then a state of equibiaxial stress will eventually be reached within the entire field. The corresponding load is obtained from eqn (92), with $\rho_b = 1$, namely

$$S = \frac{Ns}{Ns(1 - s) - 1 + \nu} \tag{98}$$

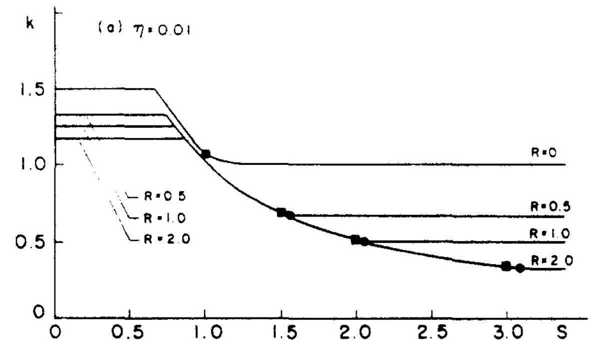
with the obvious stress concentration factor

$$k = 1 - s. \tag{99}$$

Any further increase of the load, beyond eqn (98), will just raise the magnitude of the constant stress components, $\Sigma_r = \Sigma_\theta = T$, without affecting the stress concentration, eqn (99).

Figure 3 illustrates the variation of the stress concentration factor with increasing load and for different values of parameter R. In comparison with the hole problem, Fig. 2, the R-sensitivity appears to be lower here—particularly in the intermediate loading range—probably due to the strong geometrical constraint.

The total effective strain at the inclusion is still given by eqn (50), and can be used to assess the validity of the small strain assumption. Another quantity of interest is the radial



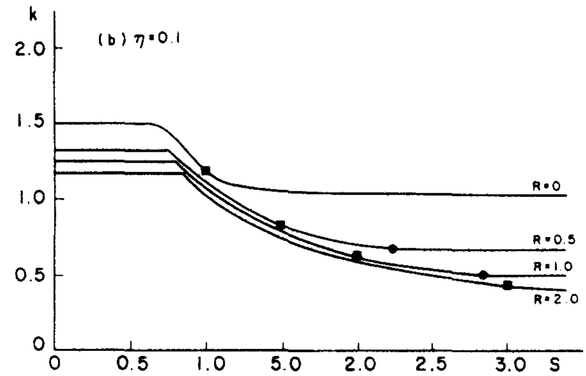


Fig. 3. Stress concentration factors at a rigid inclusion with $\nu = 1/3$. The black squares indicate complete yielding of the sheet. The black circles indicate the onset of the equibiaxial state in the entire field. The Tresca solid is described by $R = 0$.

stress concentration factor, defined as

$$k_r = \frac{\sigma_r(\rho = 1)}{\sigma_\infty} \tag{100}$$

which is important in avoiding separation between the inclusion and the sheet. A straightforward derivation gives

$$k_r = \left(\frac{1 + Ns^2}{1 - \nu s}\right)k - \left(\frac{Ns^2}{1 - \nu s}\right)\frac{1}{S} \tag{101}$$

implying that $k_r > k$.

For arbitrary hardening characteristics it is possible to obtain a differential equation similar to eqn (54). First, we rewrite the constitutive relations (72) and (73), with the aid of eqn (65), as

$$\varepsilon_r = \left(\frac{s - \nu}{s}\right)\Sigma_r + \left(\frac{\nu}{s}\right)\Sigma + \varepsilon_P \tag{102}$$

$$\varepsilon_\theta = \left(\frac{1 - \nu s}{s}\right)\Sigma_r - \left(\frac{1}{s}\right)\Sigma - s\varepsilon_P. \tag{103}$$

Next, we use the same procedure that led to eqn (54). The result reads

$$(1+s)\frac{d\Sigma_r}{d\varepsilon_{app}} = \frac{(1-s)\Sigma_r - \Sigma}{(1-s)\Sigma_r - \varepsilon_{app}} \tag{104}$$

where

$$\varepsilon_{app} = \Sigma + s^2\varepsilon_p \tag{105}$$

is the apparent total effective strain. The boundary conditions that supplement this equation are $\Sigma_r = T$ when $\Sigma = (1-s)T$ at infinity, and $(1-\nu s)\Sigma_r = \varepsilon_{app}$ when $\Sigma = kT$ at the inclusion. For the Tresca model, with $s = 0$, eqn (104) becomes an identity ($\Sigma_r \equiv \Sigma$) but it is then possible to derive an alternative equation relating Σ_θ to Σ.

At the extreme case of $R \to \infty$ and $s \to 1$, eqn (104) is reduced to $2d\Sigma_r/d\varepsilon = \Sigma/\varepsilon$ or, after integrating

$$\int_0^{kT} \frac{E_s}{E_T} d\Sigma = 2\left(\frac{\varepsilon_a}{1-\nu} - T\right). \tag{106}$$

With the Budiansky modification, eqn (56), we arrive at the equation

$$\left(\frac{2}{1-\nu}\right)(kS)^n - nkS - 2S + n - 1 = 0, \qquad S \geqslant \frac{1+\nu}{2(1-\nu)} \tag{107}$$

and asymptotically, for large S

$$k \sim (1-\nu)^{1/n} S^{-(n-1)/n}. \tag{108}$$

When $E \to \infty$ we get from eqn (104) the restricted version for a rigid/plastic material, namely

$$(1+s)\varepsilon\frac{d\sigma_r}{d\varepsilon} + (1-s)\sigma_r = \sigma_e. \tag{109}$$

However, the solution of this equation cannot be applied to the inclusion problem since the elastic terms are essential for a proper statement of the boundary condition at the inclusion. It can be used, though, for the boundary value problem where the stress σ_r is specified over the boundaries.

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